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Flows on Two-Dimensional Networks

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SUMMARY

A "two-dimensional network" G_b is here defined as an extension of an ordinary network (of the Graph Theory), whose arcs have an associate given direction, i.e. a vector $\boldsymbol{b}$ of R^2 such that $|\boldsymbol{b}|^2 = 1$.

Some concepts of the Graph Theory are extended to G_b and the path-flow decomposition of a flow on G_b is given, using the concept of bi-dimensional path.

Introduction

This paper takes his motivation from the study of the optimum conditions for the static equilibrium of planar pinned trusses with concentrated loads. A planar pinned truss is a system of rigid bars, connected by joints in such a way to obtain a rigid structure G with a plane of symmetry. External constraints are added in order to eliminate the three degrees of freedom of G in its plane. External concentrated loads, lying in the plane of the structure, are supposed to be applied at the joints.

A fundamental problem is that of evaluating the maximum loads supported by the structure at the equilibrium conditions. For different kinds of structures, with different kinds of constraints and loads, this problem is reducible to a linear program. See for example [2].

We concentrated our attention to the case where the planar pinned truss is externally statically determined and one external load is applied. Thus a network can be drawn as follows: its set of nodes $\mathcal{N}$ represents the joints and the point of convergence of the external forces (this point exists for the external equilibrium of the truss), its set of arcs $\mathcal{A}$ corresponds to the bars of the truss and the lines of action of the external forces.

For the mathematical formulation of the problem we use the following definitions.

1. First definitions

1.1. Let $G = (\mathcal{N}, \mathcal{A})$ be a connected network with n nodes, N_i, $(i = 1, 2, \ldots, n)$, and m arcs b_j, $(j = 1, 2, \ldots, m)$.

We fix a cartesian frame of reference on the plane of G and associate to each arc b_j a given direction, i.e. a vector of R^2, denoted $(\alpha_j, \beta_j)^T = \boldsymbol{b}_j$, such that $\alpha_j^2 + \beta_j^2 = 1$.

We shall call the directed network a *bi-dimensional network* G_b. By analogy the non directed network G could be called "one-dimensional". (In reference [1] it is shown that a planar truss, with all its bars parallel to each other, is equivalent to a network G). We call the *node-arc incidence matrix* of G_b the matrix

$$\mathcal{Q} = \{\boldsymbol{a}_{ij}\} \tag{1}$$

whose general entry is defined as

$$\boldsymbol{a}_{ij} = \begin{cases} (\alpha_j, \beta_j)^T = \boldsymbol{b}_j & \text{if arc } b_j \text{ "leaves" node } N_i \\ -(\alpha_j, \beta_j)^T = -\boldsymbol{b}_j & \text{if arc } b_j \text{ "enters" node } N_i \\ (0, 0)^T & \text{if arc } b_j \text{ is not incident to } N_i . \end{cases}$$

$\mathcal{Q}$ is a $2n$ by m real matrix, with two opposite non-zero vectors in each column. We shall call this property a *pseudo-unimodularity* property of the real matrix $\mathcal{Q}$, and call the set of two rows, corresponding to any node N_i, the *i-th node-row*.

1.2. Suppose that each arc b_j has a given capacity c_j, $c_j > 0$, and that two nodes of $\mathcal{N}$, say N_1 and N_n, are respectively the *source* and the *sink* for a flow (to be defined) "entering" G_b with a given direction $\boldsymbol{b}_I$.

In order to have a circulation-flow formulation of the problem, we add to $\mathcal{A}$ the *return arc* (N_n, N_1), with the direction $\boldsymbol{b}_I$, and denote G_b^* the extended network. Thus the total flow "leaving" G_b at the sink will have the same direction $\boldsymbol{b}_I$.

The incidence matrix will become a $2n$ by $(m+1)$ real matrix $\mathcal{Q}^*$, of the following form

$$\mathcal{Q}^* = \begin{array}{c|c|c|} {}_{i}\diagdown^{j} & I & 1 \quad 2 \quad \dots \quad m \\ \hline N_1 & -\begin{pmatrix}\alpha_I\\ \beta_I\end{pmatrix} & \\ N_2 & \begin{pmatrix}0\\ 0\end{pmatrix} & \mathcal{Q} \\ \vdots & \vdots & \\ N_n & \begin{pmatrix}\alpha_I\\ \beta_I\end{pmatrix} & \end{array} \tag{2}$$

where $\mathcal{Q}$ is the incidence matrix defined in (1).

Denote $\omega(N_i)$ the i-th node row of $\mathcal{Q}^*$.

We define a *flow* $\boldsymbol{f}$ *on* G_b^* (or G_b) any vector of R^{m+1}, $\boldsymbol{f} = (f_I, f_1, \dots, f_m)$ *with* $f_I \neq 0$, such that

$$\mathcal{Q}^* \boldsymbol{f}^T = \boldsymbol{0}. \tag{3}$$

$f_I =$ *value* of the flow $\boldsymbol{f}$. $\boldsymbol{f}$ is *feasible* if

$$-c_j \leqq f_j \leqq c_j \qquad j = I, 1, 2, \dots, m \ \ (\text{for } j = I \text{ suppose } c_I = \infty). \tag{4}$$

An obvious feasible flow is $\boldsymbol{f} = \boldsymbol{0}$.

The problem of finding the maximum value of f_I is the analogue of the max flow problem for G, [3]. The simplex method can evaluate f_I max, since "maximize f_I such that (3) and (4) hold" is a linear program.

1.3. In what follows we shall try to perform the path-flow decomposition of a flow $\boldsymbol{f}$ on G_b, and thus we need a path of G_b.

2. Bi-dimensional Paths

2.1. A *bi-dimensional path* of G_b (or path, when no confusion with ordinary paths of the Graph Theory may arise) from *source to sink* is a subset P of arcs of G_b, connecting N_1 to N_n, satisfying the rules given below. (These rules have a familiar meaning to the Mechanical and Civil Engineers).

Since we will work constantly on G_b^*, in the following we shall call improperly a *path of* G_b^* any set $P \cup \{b_I\}$, where P is a path of G_b, instead of a *circuit of* G_b^* *containing the return arc.*

We denote $\mathcal{N}(P)$ the set of nodes incident to the arcs of P.

2.2. Rules for constructing a path P of G_b^*.

Start : The return arc enter N_1; $b_I \in P$.

Rule (*a*): If one arc b_j of P enters the node N_i and the set, $\omega(N_i)$, of arcs incident to N_i contains either

(1) one arc $b_k // b_j$, $b_k \notin P$, or

(2) two arcs $b_{k_1} \# b_{k_2} \# b_j$, b_{k_1} and $b_{k_2} \notin P$,
then $N_i \in \mathcal{N}(P)$. The path leaves N_j through b_k or both b_{k_1} and b_{k_2}.

Rule (*b*): If two arcs (or more) not (pairwise) parallel of P enter N_i and $\omega(N_i)$ contains either
(1) one arc parallel to none of them, $b_k \notin P$, or
(2) two arcs $b_{k_1} \# b_{k_2} \# b_j, \forall b_j \ni \begin{cases} b_J \in P, \\ b_J \in \omega(N_i) \end{cases} b_{k_1}$ and $b_{k_2} \notin P$,
then $N_i \in \mathcal{N}(P)$ and the path leaves N_i through b_k or b_{k_1} and b_{k_2}.

If $N_i = N_n$, the cases (a.1) and (b.1) can hold in particular, but then the condition $b_k \notin P$ must be dropped when $b_k = b_I$.

Rule (*c*): If two parallel arcs of P enter N_i, then $N_i \in \mathcal{N}(P)$ and the corresponding branch of the path ends at N_i.

Rule (*d*): If several arcs enter N_i and $\omega(N_i) - P = \phi$, then $N_i \in \mathcal{N}(P)$ and the corresponding branch of P ends at N_i.

Stop : When none of the rules can be applied or when N_n belongs to $\mathcal{N}(P)$ and all the nodes reached by arcs of P belong to $\mathcal{N}(P)$.

Any node can be reached by the path more than once.

Example 1: Suppose $b_I // b_7 // b_8$, $b_2 // b_5$. We have

$$P_1 = \{b_I, b_7, b_8\} \text{ with } \mathcal{N}(P_1) = \{N_1, N_3, N_n\},$$

$$P_2 = \{b_I, b_1, b_4, b_2, b_3, b_5, b_6\} \text{ with } \mathcal{N}(P_2) = \{N_1, N_2, N_4, N_3, N_n\}.$$

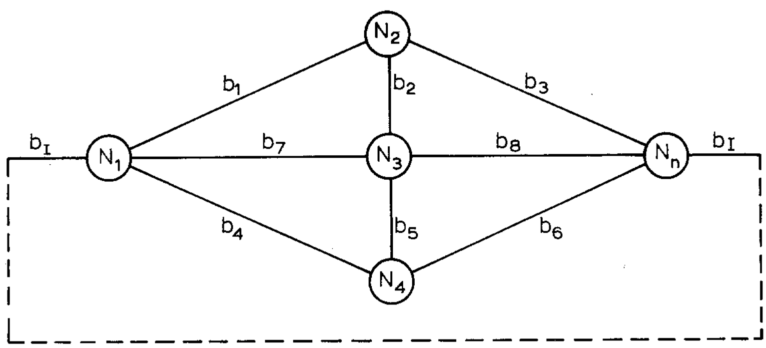

Figure 1.

2.3. We shall call a *path-flow* $\boldsymbol{f}^{(k)} = \{f_j^{(k)}\}$ any flow whose support is a path. The support, $\|\boldsymbol{f}\|$, of a vector $\boldsymbol{f}$ on $\mathscr{A}$ is usually defined as the subset of arcs of $\mathscr{A}$ for which $f_j \neq 0$.

Property: *The support of a path-flow is minimal.*

Indeed, let's denote $\mathcal{N}_{(a)}$ the set of nodes of P for which the rule (a) or (c) holds; $\mathcal{N}_{(a)} \neq \phi$. Clearly no arc $b_j \in P \cap \omega(\mathcal{N}_{(a)})$ can be suppressed without breaking the conservation law at some node $N_i \in \mathcal{N}_{(a)}$. If $\mathcal{N}'_{(b)}$ is the set of nodes of P, for which rule (b) holds, adjacent to the nodes of $\mathcal{N}_{(a)}$, then no arc of $(\mathcal{N}_{(a)}, \mathcal{N}'_{(b)})$ can be suppressed because of the minimality of the number of arcs incident to $\mathcal{N}_{(a)}$. Thus the net flow entering N_i ($\forall N_i \in \mathcal{N}'_{(b)}$) from $\mathcal{N}_{(a)}$ has a fixed direction, say $\boldsymbol{b}_i^*$, and the vector conservation equation at N_i for the path-flow $\boldsymbol{f}$, i.e.

$$\omega(N_i) \times \boldsymbol{f} = \boldsymbol{0} \qquad N_i \in \mathcal{N}'_{(b)}$$

can be written as

$$f_i^* \boldsymbol{b}_i^* + f_k \boldsymbol{b}_k = \boldsymbol{0} \quad \text{if} \quad b_i^* // b_k$$

or

$$f_i^* \boldsymbol{b}_i^* + f_{k_1} \boldsymbol{b}_{k_1} + f_{k_2} \boldsymbol{b}_{k_2} = \boldsymbol{0} \quad \text{if} \quad b_i^* \# b_{k_1} \# b_{k_2}.$$

If $f_i^* \neq 0$, no leaving arc can be suppressed for the equilibrium of N_i; if $f_i^* = 0$ then either

$$f_k = 0 \Rightarrow b_k \notin \|f\|, \quad \text{or}$$

$$f_{k_1} = f_{k_2} = 0 \Rightarrow b_{k_1}, b_{k_2} \notin \|f\| \quad \text{or}$$

$$b_{k_1} /\!/ b_{k_2} \quad \text{against the definition of path.}$$

By the same argument we can prove that it is not possible to suppress any arc of the path incident to nodes of $\mathcal{N}''_{(b)}$, $\mathcal{N}''_{(b)}$ being the set of the nodes of P adjacent in P to $\mathcal{N}_{(a)}$ and $\mathcal{N}'_{(b)}$, without breaking the equilibrium of some node or getting a contradiction. In particular the suppression of all the arcs of P, incident to any node N_i, does not affect the equilibrium of this node, but destroys that of the adjacent ones.

Because of the property of minimality, we call *elementary* path any path being the support of a flow.

An elementary path, if it exists, can be constructed by the methods of Graphic Statics, like the Maxwell–Cremona method [4], starting with a given arbitrary value for f_I.

We remark that the networks G_b corresponding to some kind of isostatic planar pinned trusses, externally statically determined, are themselves elementary paths, although this is not true in general for every kind of statically determined structure, as shown in Fig. 3; moreover not every elementary path corresponds to an isostatic structure.

2.4. Some analytical considerations.

If P is a path from source to sink of G_b^* (or G_b), denote $\mathcal{Q}_P^*$ the submatrix of $\mathcal{Q}^* \equiv \mathcal{Q}^*_{\mathcal{A}(G)}$ corresponding to the arcs of P, and $\mathcal{Q}_P$ the corresponding submatrix of $\mathcal{Q}$. Then we say that

Theorem 1: *P is an elementary path iff rank $(\mathcal{Q}_P^*) =$ rank $(\mathcal{Q}_P)$ and the columns of $\mathcal{Q}_P$ are a non-degenerate basis for $\mathcal{Q}_P^*$.*

Proof: If P is an elementary path, then by definition there is a flow f such that $\|f\| = P$. By mechanical considerations, f is uniquely determined by any (non-zero) value of f_I: indeed at any node of P a known vector is decomposed into two given non-parallel directions or into a parallel one, and this operation has a unique solution on the plane. Analytically this can be done only if rank $(\mathcal{Q}_P^*) =$ rank $(\mathcal{Q}_P) = |P| - 1$, otherwise a flow exists whose support is properly contained in P.

Conversely, if P is a path and rank $(\mathcal{Q}_P^*) =$ rank $(\mathcal{Q}_P) = |P| - 1$, then clearly the column of b_I is a linear combination of columns of $\mathcal{Q}_P$, so that the system of linear equations $\mathcal{Q}_P^* f_P^T = \mathbf{0}$ has a unique non-zero solution f_P, for any fixed non-zero value of f_I. And it is $\|f\| \notin P$, because $\mathcal{Q}_P$ is a non-degenerate basis for $\mathcal{Q}_P^*$. Thus the vector f, $f = f_P$ on P and 0 on $\mathcal{A} - P$, is a flow of support P, i.e. $P =$ elementary path. Q.E.D.

It follows directly that

Corollary 1: *If $P = \mathcal{A}(G)$ is an elementary path, then it is the unique path of G_b.*

The converse is clearly not true.

Example 2: The unique path $P = \mathcal{A}(G)$ of fig. 2 is not elementary if, for example, $f_2 b_2 + f_5 b_5 = f^* b^* \# b_7$, where f_2 and f_5 are uniquely determined from the conservation equations at N_2 and N_4 respectively.

Corollary 2: *If P is an elementary path of G_b and $r(\mathcal{Q}^*) = r(\mathcal{Q}) = m$, then P is the unique elementary path.*

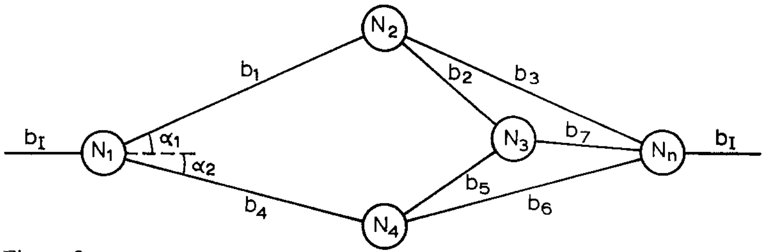

Figure 2.

Proof: If P is an elementary path, then there exists a set of columns, $\{c_{j_k}\}$, of $\mathscr{Q}^*$ such that

$$\sum_{\substack{k=1\\k\neq I}}^{|P|-1} x_k c_{j_k} = c_I \, . \tag{5}$$

If $r(\mathscr{Q}^*) = r(\mathscr{Q}) = m$ then the equation (6),

$$\sum_{\substack{j=1\\j\neq I}}^{m} y_j c_j = 0 \, , \tag{6}$$

holds only if $y_j = 0 \forall j$, and a set of not all zero coefficients z_j exists, such that

$$\sum_{j=1}^{m} z_j c_j = c_I \, . \tag{7}$$

If it exists a solution of (7), whose support is different from P, then by comparing (5) and (7) we get a contradiction to (6). Q.E.D.

Remarks to the Corollary 2:

(a) It can be $P \subset \mathscr{A}(G)$.

(b) The converse of the Corollary is not true for every network, i.e. the existence of a unique elementary path P does not exclude the possible existence of a flow f on G_b such that $\| f \| \subseteq \mathscr{A}(G) - P$. For example, the following network does not contain any path from source (N_1) to sink (N_n) although a solution of $\mathscr{Q}^* f^T = \mathbf{0}$, $f_I \neq 0$, exists.

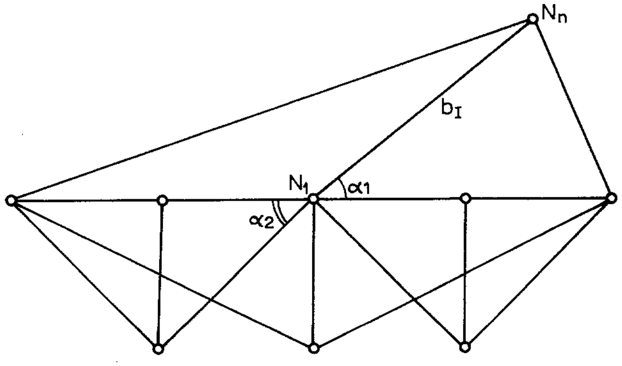

Figure 3.

Assumption. For the following, we will assume that the converse of the Corollary 2 holds for the networks we are dealing with.

We let untouched the question of characterizing such networks.

3. Independent Elementary Paths

3.1. We call *independent set* of elementary paths any set $\mathscr{I}_P = \{P_1, \ldots, P_l\}$ of distinct paths, none of which is properly contained in any union of others.

We call a $\mathscr{P}$*-set* an $\mathscr{I}_P$-set such that every elementary path is a proper subset of the union of its elements.

A $\mathscr{P}$-set is maximal (i.e. not properly contained in any other $\mathscr{I}_P$-set) by definition.

Property 1: *A $\mathscr{P}$-set exists, under our assumption.*

Indeed: let b_j be an arc not belonging to any path of a maximal independent set $\mathscr{I}_P$. Then, either b_j does not belong to any path of G_b, or $b_j \in P$. P is not independent from the elements of $\mathscr{I}_P$, otherwise $\mathscr{I}_P \subset (\mathscr{I}_P \cup \{P\})$ against the hypothesis of maximality; but $\mathscr{I}_P \cup \{P\}$ is a collection of paths containing as a proper subset another independent set, the union of whose elements contains every elementary path contained in unions of elements of $\mathscr{I}_P$. We can repeat the procedure until an arc b_j is found belonging to some path. Q.E.D.

For a selected $\mathscr{P}$-set, any arc of G_b can be classified as
(a) *independent arc*, b_{ik}, if it belongs exactly to one path of $\mathscr{P}$;
(b) *common arc*, b_c, if it belongs to more than one path of $\mathscr{P}$;
(c) *idle arc*, b_0, if it does not belong to any path.
(Here and in the following, path means elementary path).

3.2. An *independent set of arcs* $\mathscr{I}_b$ is a family of independent arcs $b_{i_1}, b_{i_2}, \ldots, b_{i_l}$, $l = |\mathscr{P}|$, such that $b_{i_j} \in P_j \in \mathscr{P}$, $b_{i_{j'}} \in P_{j'} \in \mathscr{P}$, $P_j \neq P_{j'}$ for $j \neq j'$.

A *bi-dimensional cut-set* $\mathscr{C}$, separating the source from the sink, is a minimal set of arcs whose removal disconnects every path.

3.3. We are now going to construct a family of paths, that we will use in the path-flow decomposition of f.

Take a $\mathscr{P}$-set and an independent set of arcs $\mathscr{I}_b$. Then consider the network $\bar{G}_1 = (\mathscr{N}, \mathscr{A} - \mathscr{I}_b = \mathscr{A}(\bar{G}_1))$. If $\bar{G}_1$ does not have any path, then $\mathscr{I}_b = \mathscr{C}$; if it does, then take a $\mathscr{P}$-set of $\bar{G}_1$, say $\mathscr{P}_1$, and pick an independent set of it, $\mathscr{I}_{1b}$. If the network $\bar{G}_2 = (\mathscr{N}, \mathscr{A} - \mathscr{I}_b - \mathscr{I}_{1b})$ does not have any path, then $(\mathscr{I}_b \cup \mathscr{I}_{1b}) \supseteq \mathscr{C}$. Indeed: if no path is contained in $\mathscr{A}(\bar{G}_1)$, then any path P of G_b, $P \neq P_k$, $k = 1, \ldots, l$, if it exists, must contain at least one arc of $\mathscr{I}_b$, since it cannot be properly contained in any $P_k \in \mathscr{P}$. If no path is contained in $\mathscr{A}(\bar{G}_2)$, then $\mathscr{I}_{1b}$ is a cut-set for $\bar{G}_1$; $\mathscr{I}_b \cup \mathscr{I}_{1b}$ is not necessarily a cut-set for G_b, because all the arcs of $\mathscr{I}_{1b}$ are common between paths of $\mathscr{P}$ and $\mathscr{P}_1$.

If $\bar{G}_2$ has a path, we can proceed until we find a $\bar{G}_k$ which does not have any.

Call $\mathscr{P}_B$ the family of paths given by the union of the $\mathscr{P}_i$-sets, i.e. $\mathscr{P}_B = \{\mathscr{P} = \mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_{k-1}\}$, and denote its elements $P_1, \ldots, P_{l_0}, P_{l_0+1}, \ldots, P_L$.

We can consider the arc-path incidence matrix $\{p_{jk}\}$, by defining $p_{jk} = 1$ if $b_j \in P_k$, $p_{jk} = 0$ if $b_j \notin P_k, \forall k = 1, \ldots, L$. It is clear that, if $f^{(1)}, f^{(2)}, \ldots, f^{(t)}$, $t \leqq L$, are path-flows on G_b, of values $f_I^{(1)}, f_I^{(2)}, \ldots, f_I^{(t)}$ respectively and constructed in such a way that the double inequality is satisfied

$$-c_j \leqq \sum_{k=1}^{t} p_{jk} f_j^{(k)} \leqq c_j, \quad \forall j, \tag{8}$$

then we have a feasible flow f on G_b, of general component

$$f_j = \sum_{k=1}^{t} p_{jk} f_j^{(k)}$$

and of value $f_I = \sum_{k=1}^{t} f_I^{(k)}$.

We want to show now the converse, i.e. that *any feasible flow can be decomposed into path-flows* $f^{(k)}$, such that $f = \Sigma_k f^{(k)}$. Let f be a feasible flow on G_b, of value f_I, and $\mathscr{P}_B$ any family of paths, selected as above.

Step 1: Take any elementary path of $\mathscr{P}_0$, properly contained into $\|f\|$, say $P_1 \subseteq \|f\|$. If such a path cannot be found, then go to Step 2. If Step 2 cannot work on the paths of $\mathscr{P}_0$, then go back to the Step 1 with $\mathscr{P} \Rightarrow \mathscr{P}_1$, and so on until one of the two steps can be applied.

If $P_1 = \|f\|$, then our claim is proved.

If not, pick an independent arc of P_1, say b_{j_1}, and set $f_{j_1}^{(1)} = f_{j_1}$. ("Independent" means relative to the $\mathscr{P}_i$-set we are dealing with). Then, by Theorem 1, we get a path-flow $f^{(1)}$ of value $f_I^{(1)}$.

We do not care about the feasibility of the individual path-flows, since they "flow" simultaneously on G_b.

Then take the reduced flow $f_1 = f - f^{(1)}$ and, since $\|f_1\| \neq \phi$, search for another path P_2 of $\mathscr{P}_0$ such that either

(a) $P_2 \subseteq \|f_1\|$, or

(b) $P_2 \cap \|f_1\| \neq \phi$, $P_2 - \|f_1\| \neq \phi$.

If (a) occurs, repeat Step 1 until possible, taking in the order the paths of $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_{k-1}$.
If (b) occurs, then go to Step 2.

Step 2: If no elementary path exists, properly contained in $\|f\|$, (as in Example 4, where the dotted arc does not belong to $\|f\|$), then consider any elementary path $P_k \in \mathscr{P}_0$ such that $P_k \cap \|f\| \neq \phi$.

(a) If an independent arc b_{i_k} of P_k belongs to $\|f\|$, then, by Theorem 1, there is a "partial flow" flowing along P_k. Note that the "total" arc-flow f_j on some common arc of P_k can nevertheless be zero.

We can calculate $f^{(k)}$, such that $f_{i_k}^{(k)} = f_{i_k}$. The arcs of $P_k - \|f\|$ will belong to the support of the residual flow $f' = f - f^{(k)}$, and we can start again with f'.

(b) If no independent arc of P_k carries flow, then, by Theorem 1, no arc of P_k is support of a "partial" arc-flow.

Then we have to try again, taking into account the successive $\mathscr{P}_i$-sets, until case (a) will occur.

Example 4:

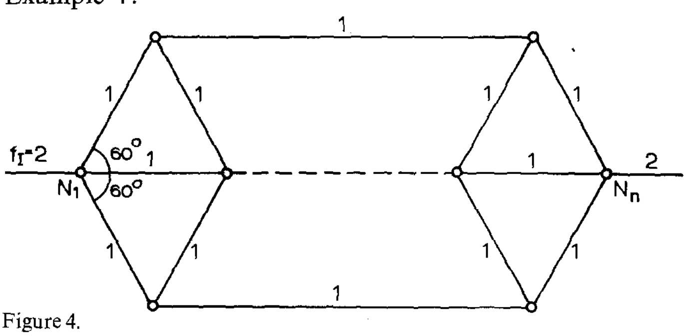

Figure 4.

By assumption we excluded the case where a flow exists on G_b, but no paths from source to sink can be found such that $P \cap \|f\| \neq \phi$ (as in Example 4 if the dotted arc does not belong to G); thus when all the paths of $\mathscr{P}_B$ will be examined no arc-flow can be left, otherwise either the conservation equation doesnot hold at some node, or a solution exists of $\mathscr{Q}^* f^T = \mathbf{0}$, with $f_I = 0$, which is not a flow.

3.4. We claim that, when no more residual arc-flows can be found, then we have

$$f^* = \sum_{k=1}^{L} f^{(k)} = f \qquad (\|f\| \subseteq \bigcup_k \|f^{(k)}\|), \tag{9}$$

with

$$-c_j \leqq f_j = \sum_k p_{jk} f_j^{(k)} \leqq c_j \qquad \forall b_j \in \| f \| , \tag{10}$$

$$f_j = \sum_k p_{jk} f_j^{(k)} = 0 . \qquad \forall b_j \notin \| f \| . \tag{11}$$

Proof: (10) is true, by construction, for the independent arcs of $\mathscr{I}_b$. Denote them $b_1, b_2, \ldots, b_{l_0}$; then for every elementary path of $\mathscr{P}_0$ we have

$$\sum_{\substack{j \neq k \\ j=1}}^{|P_k| = l_0} f_j^{(k)} c_j^{(k)} + f_k c_k^{(k)} = 0 \qquad \forall k = 1, \ldots, l_0, \tag{12}$$

where $c_j^{(k)}$ denotes the column-vector of $\mathscr{Q}^*$ corresponding to the arc b_j of the path P_k.

Since for the flow f we have

$$f_1 c_1^{(1)} + f_2 c_2^{(2)} + \ldots + f_{l_0} c_{l_0}^{(l_0)} + \sum_{j > l_0}^{m+1} f_j c_j = 0 , \tag{13}$$

then comparing (13) with (12), after summing those equations over k, we get

$$\sum_{j > l_0}^{m+1} \left(f_j - \sum_k f_j^{(k)} \right) c_j = 0 . \tag{14}$$

If $\mathscr{P}_0 = \mathscr{P}_B$, then (14) holds only if $(f_j - \Sigma_k f_j^{(k)}) = 0, \forall j$.

If $\mathscr{P}_0 \subset \mathscr{P}_B$, then the arcs of $\mathscr{A}$, which do not belong to any path of $\mathscr{P}_B - \mathscr{P}_0$, carry zero-flow, so that it is $(f_j - \Sigma_k f_j^{(k)}) = 0, \forall b_j$ of $\mathscr{A} - \mathscr{I}_b - \cup_{P_k \in \mathscr{P}_B - \mathscr{P}_0} P_k$.

Thus the flow $\sum_{k=1}^{l_0} f^{(k)}$ has feasible values, with the same arc-flows of f, on all the arcs of

$$\mathscr{A}(G) - \bigcup_{P_k \in \mathscr{P}_B - \mathscr{P}_0} P_k = \mathscr{A} - \mathscr{A}_0, \quad \text{and } f', \quad f' = f - \sum_{k=1}^{l_0} f^{(k)} ,$$

is a flow of support strictly contained in $\mathscr{A} - \mathscr{A}_0$.

We get the decomposition of f' along the paths of $\mathscr{P}_1$ and, if $\mathscr{P}_B = \mathscr{P}_0 \cup \mathscr{P}_1$, then the equation.

$$\sum_{j = l_0 + l_1 + 1}^{|\mathscr{A} - \mathscr{A}_0|} \left(\sum_{k=1}^{l_1 = |\mathscr{P}_1|} f_j'^{(k)} - f_j' \right) c_j = 0$$

can hold only with zero coefficients, since by construction it is

$$\sum_{k=1}^{l_1} f_j'^{(k)} = f_j' \qquad \forall b_j \in \mathscr{I}_{1b} ,$$

and so on. When the last path-flow decomposition for the residual flow (call it again f') is done, the comparison (15),

$$\sum_{j > l_0 + \ldots + l_{k-1}} \left(\sum_{t=1}^{l_{k-1} = |\mathscr{P}_{k-1}|} f_j'^{(t)} - f_j' \right) c_j = 0 , \tag{15}$$

implies all the coefficients to be zero; if not we get a contradiction to our initial assumptions. In particular it must be

$$\sum_{k=1}^{L} f_I^{(k)} = f_I ,$$

otherwise the conservation equation is not satisfied at the source (sink). Of course the preceding decomposition is not unique, depending, for the selected $\mathscr{P}_i$, on the choice of the independent sets.

4. Conclusion

In this paper the notion of two-dimensional network G_b and two-dimensional path have been given, and the path-flow decomposition of a flow on G_b has been discussed. As in the analogous problem for a one-dimensional network, this operation is the first step to the solution of the maximum flow problem: indeed a two-dimensional cut-set is properly contained in the union of the $\mathscr{I}_{ib}$ sets we found at 3.3.

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